NP-Complete and Approximation Algorithms

Polynomial Algorithms

- Problems encountered so far are polynomial time algorithms
- The worst-case running time of most of these algorithms is $O(n^k)$ time, for some constant k.
- All problems cannot be solved in polynomial time
- There are problems that cannot be solved at all – Unsolvable
- There are problems that can be solved but not in $O(n^k)$ time for some constant time.
- Problems that are solvable in polynomial time by polynomial-time algorithms are said to be tractable (or easy or efficient).
- Problems that require superpolynomial time are said to be intractable or hard.

P and NP

- Class P problems are solvable in polynomial time.
- Class NP problems are verifiable in polynomial time.
  - For example, given a problem, we can verify the solution in polynomial time
- Any problem in P is also in NP
- P $\subseteq$NP
- It is NOT KNOWN whether P is a proper subset of NP.

NP

- **What are NP-complete Problems?**
  A problem is said to be NP-Complete if it is as 'hard' as any problem in NP.
- No polynomial time algorithm has yet been discovered for an NP-Complete problem.
- However, it has not been proven that NO polynomial time algorithm can exist for an NP-Complete problem.
- This problem was first posed by Cook in 1971.
- The issue of, P=NP or P$\neq$ NP is an open research problem

Examples of NP-Complete problems

- Shortest vs. longest simple paths
- Finding the shortest paths from a single source in a directed graph G =(V,E) can be completed in O(V E) time. Even with negative edge weights.

However, finding the longest simple path between two vertices is NP-complete. It is NP-Complete even if each of edge weights is equal to one.

- **An Euler tour of a connected directed graph G=(V,E), can be completed in O(E) time.**

However, the Hamiltonian Cycle is NP-Complete.

The traveling salesman problem is a variation of the Hamiltonian cycle.

Polynomial Time Reductions

- Decision Problems: Problems whose answer is yes or no!!
  Most problems can be converted to decision problems.
- Language recognition problem is a decision problem.
- Suppose L $\subseteq$U is the set of all inputs for which the answer to the problem is yes –
  - **U is the input space and L is the language that returns a 'true' or 'yes'**
  - **S = $(x1+x2+\underline{x3})\cdot(\underline{x1}+x2+x3)\cdot(\underline{x1}+\underline{x2}+x3)$**
- L is called the language corresponding to the problem (Turing machines).

  The terms language and problem are used interchangeably.
- Given a problem, with an input language X,
- Now the decision problem can be defined as the problem to recognize whether or not X belongs to L.

Reductions Contd.

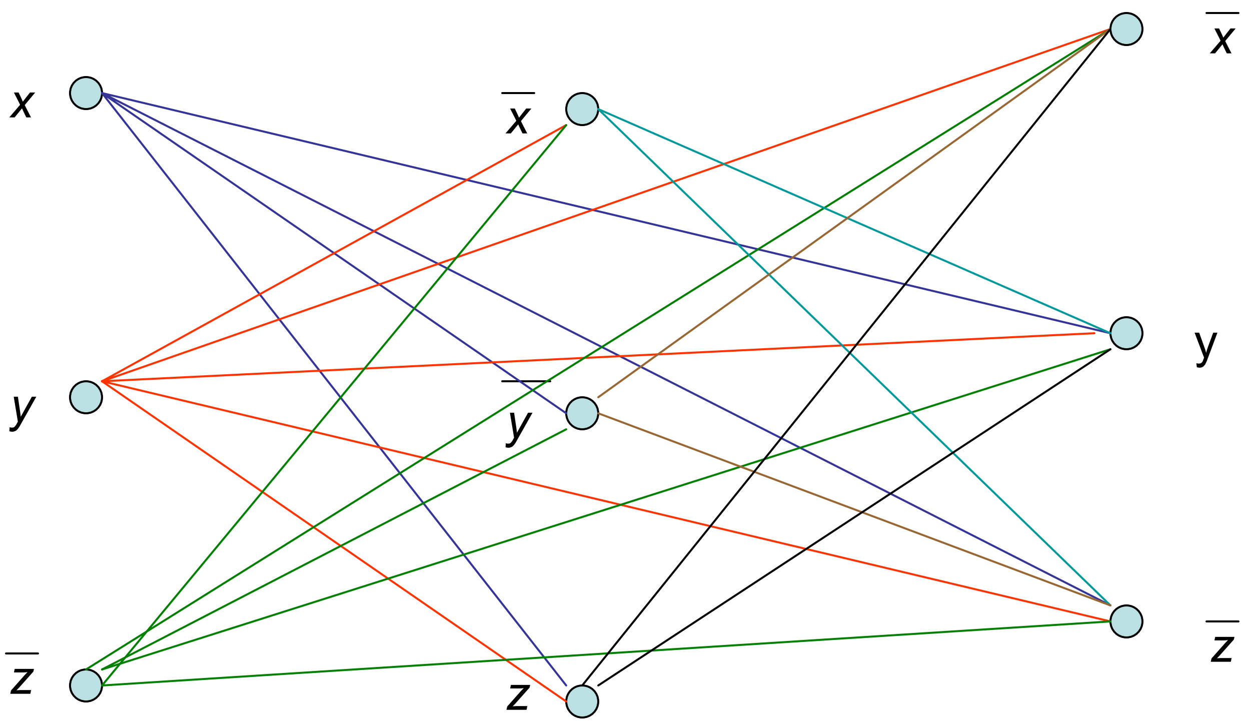
- Definition: Let L1 and L2 be two languages from input spaces U1 and U2. We say that L1 is polynomially reducible to L2 if there exists a polynomial-time algorithm that converts each input $u1 \in U1$ to another input $u2 \in U2$ such that $u1 \in L1$ iff $u2 \in L2$.
- The algorithm is polynomial in the size of the input u1.
- If we have an algorithm for L2 then we can compose the two algorithms to produce an algorithm for L1.
- If L1 is polynomially reducible to L2 and there is a polynomial-time algorithm for L2, then there is a polynomial time algorithm for L1.
- Reducibility is not symmetric
- L1 is polynomially reducible to L2 does not imply L2 is polynomially reducible to L1

The Satisfiability (SAT) Problem

- S – Boolean expression in Conjunctive Normal Form (CNF) (Product (AND) of Sums (ORs)
- For example $S = (x1+x2+\underline{x3})\cdot(\ \underline{x1}+x2+x3)\cdot(\underline{x1}+\underline{x2}+x3)$
- The SAT problem is to determine whether a given Boolean expression is Satisfiable (without necessarily finding a satisfying assignment)
- We can guess a truth assignment and check that it satisfies the expression in polynomial time.
- SAT is NP hard
- A Turing machine and all of its operations on a given input can be described by a Boolean Expression. The expression will be satisfiable iff the Turing machine will terminate at an accepting state for the given input.
- http://www.nada.kth.se/~viggo/problemlist/compendium.html

Clique Problem

$$(x + y + \overline{z}) \bullet (x + \overline{y} + z) \bullet (\overline{x} + y + \overline{z})$$



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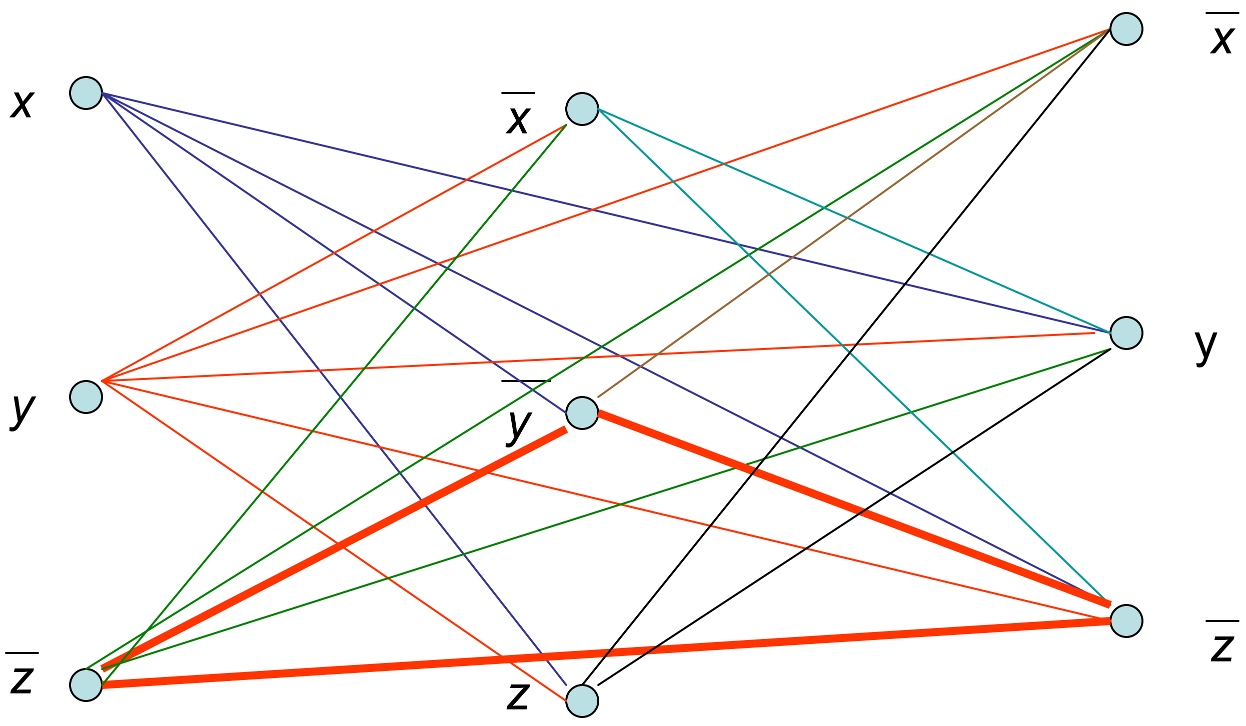

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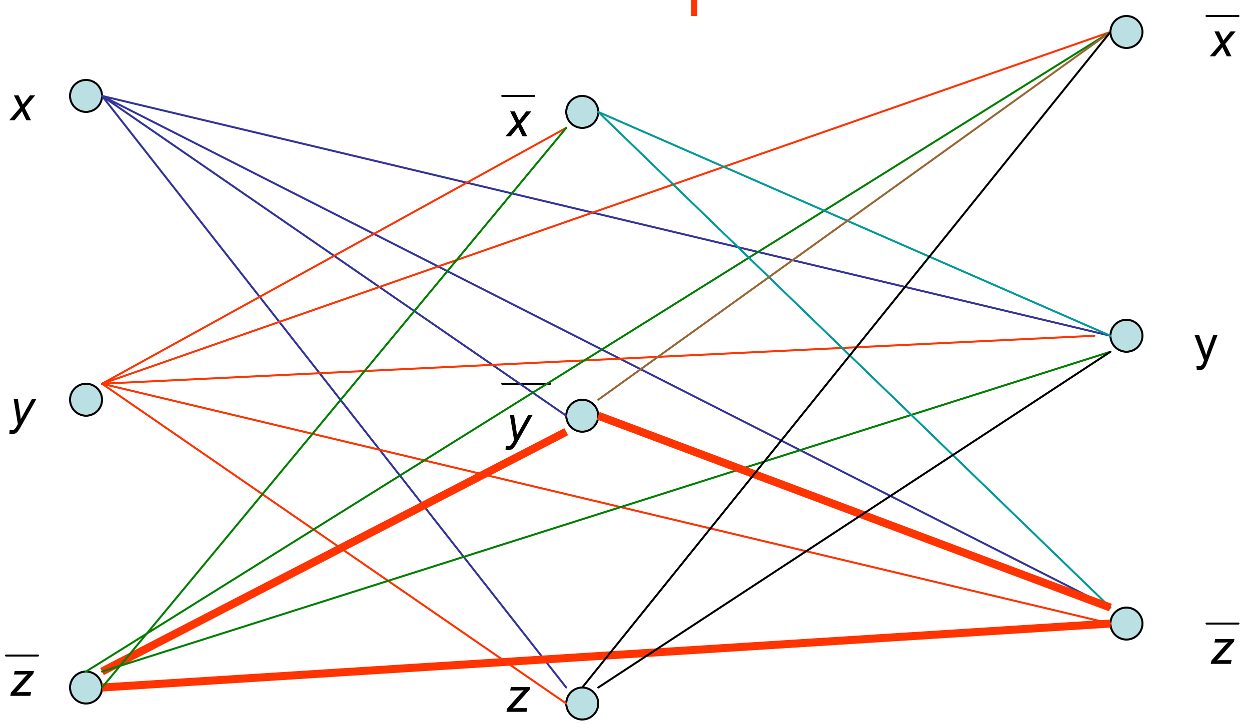

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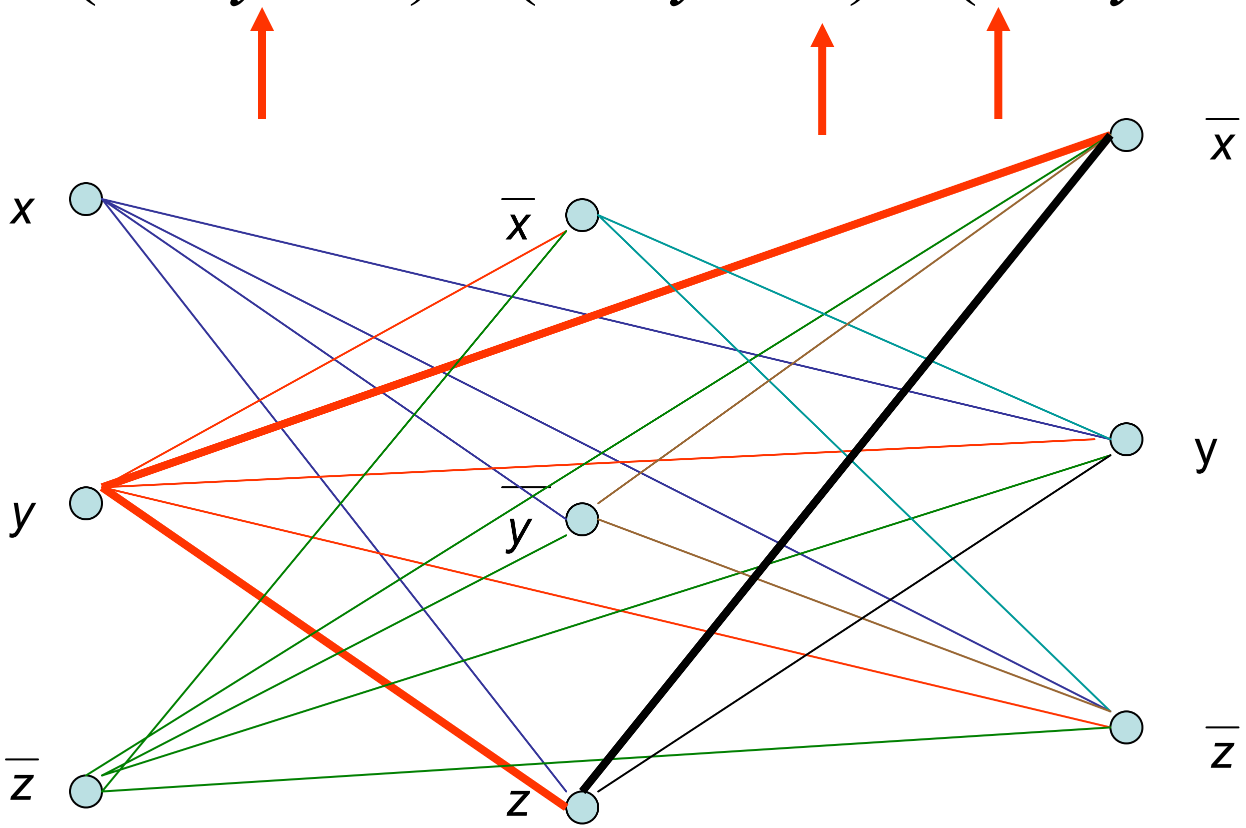

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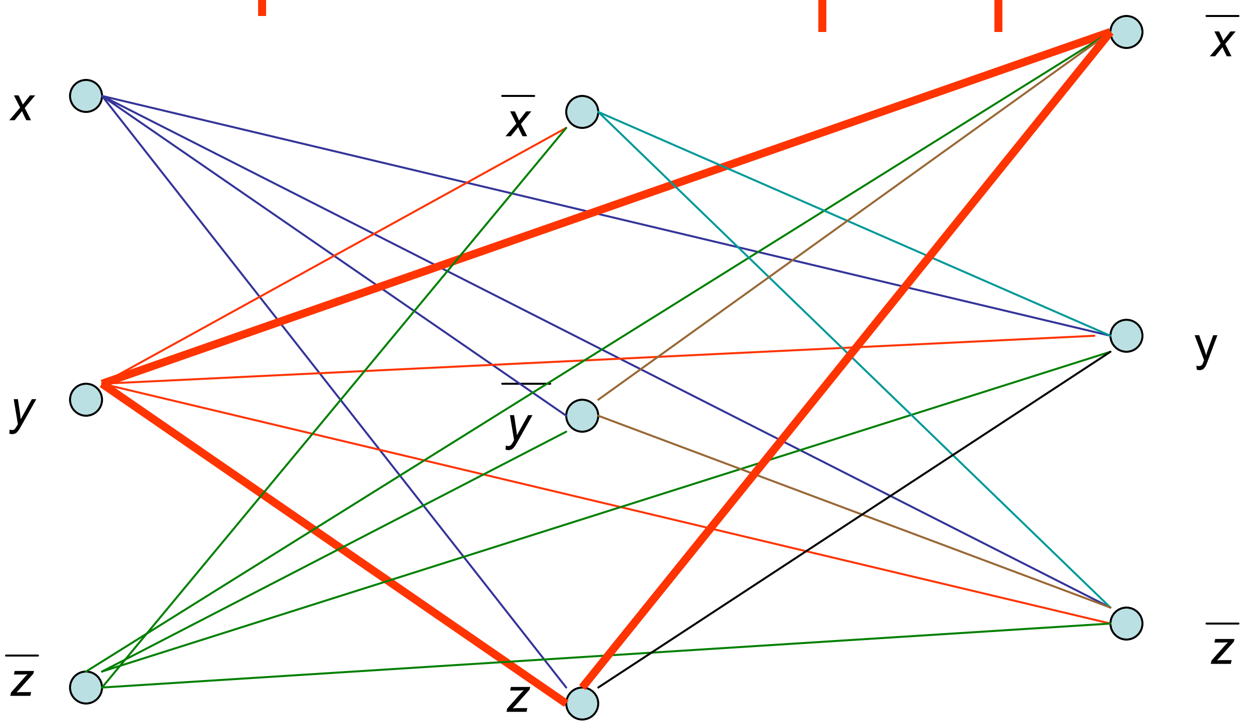
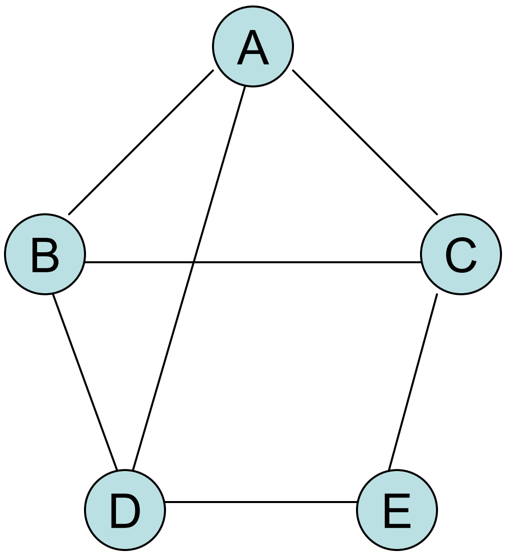


Vertex Cover problem

A vertex cover of G = (V,E) is a set of vertices such that every edge in E is incident to at least one of the vertices in the vertex cover.



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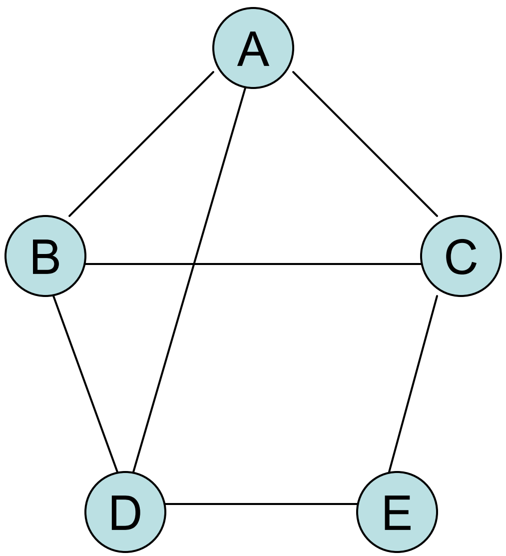
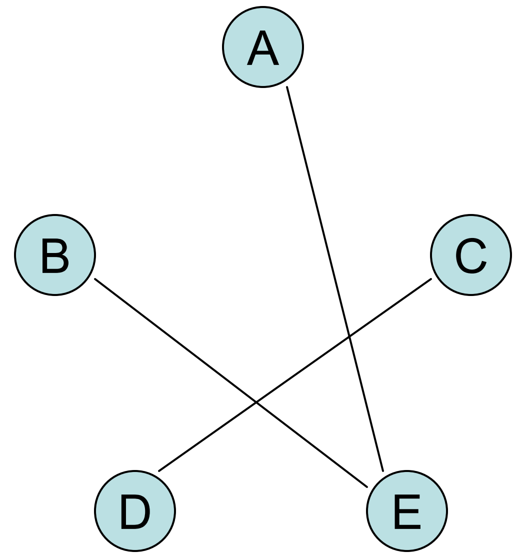


G



G'

complement of G

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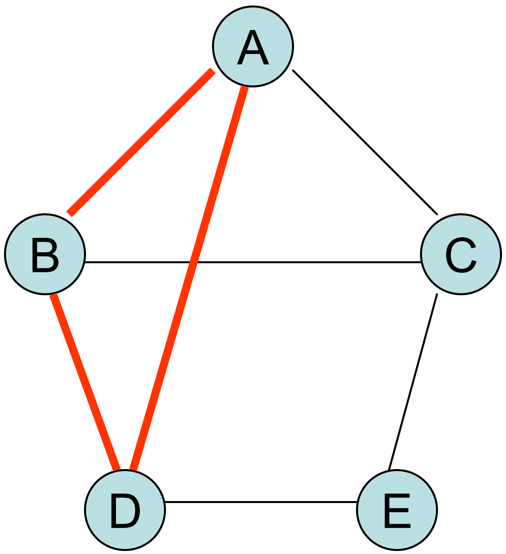

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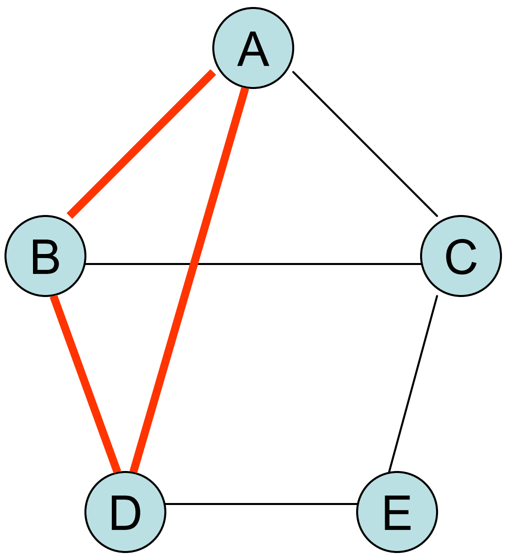

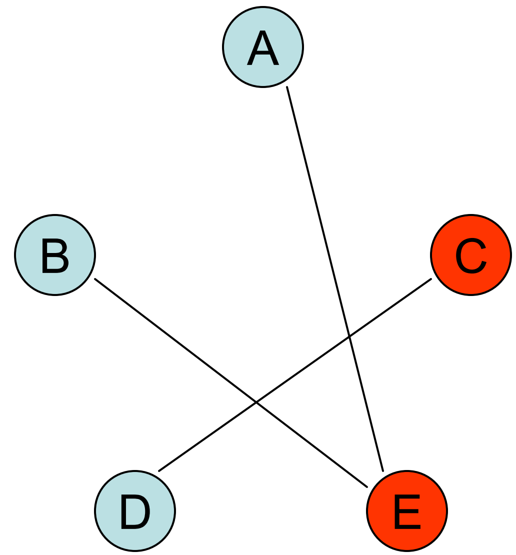

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G' has a minimum vertex cover of size k if and only if G has a maximum clique of size $|V|$-k

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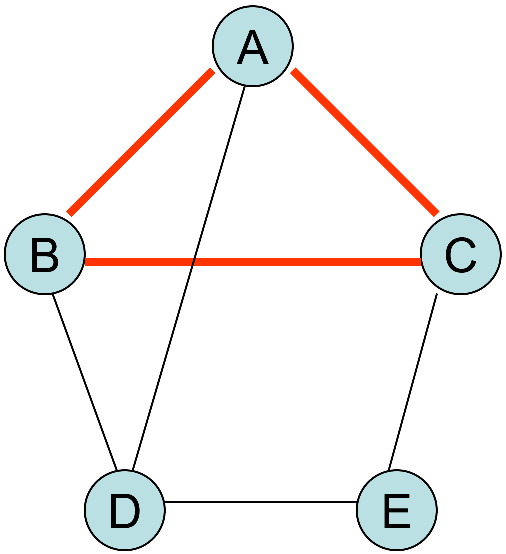
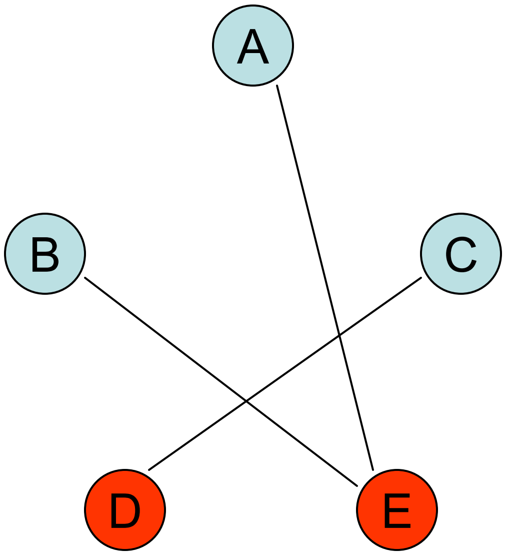


G



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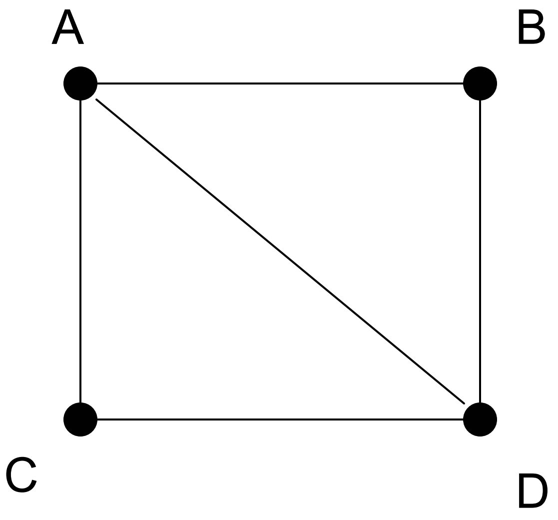
Dominating Set

- G = (V,E) is an undirected graph. A dominating set **D** of G is a set of vertices (a subset of V) such that every vertex of G is either in **D** or is adjacent to at least one vertex from **D**.



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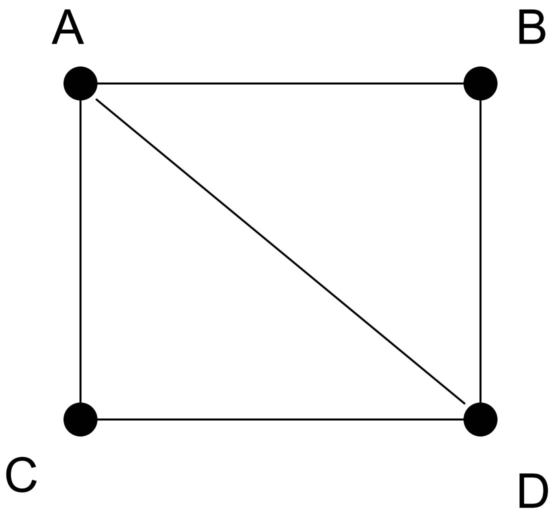

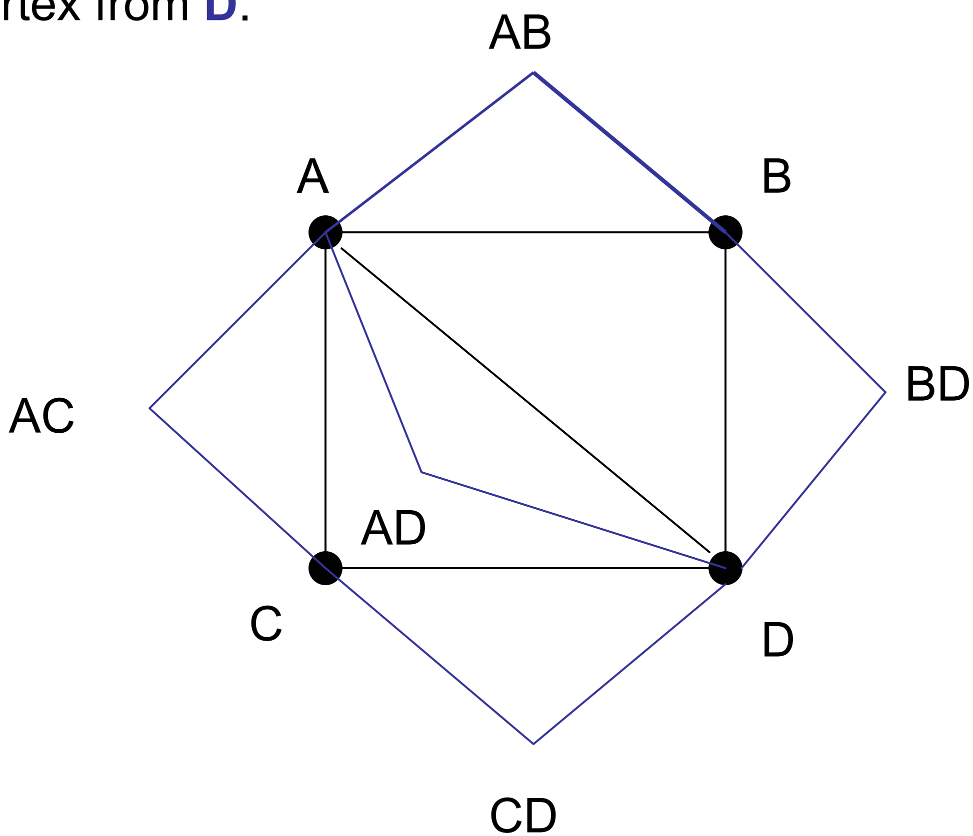

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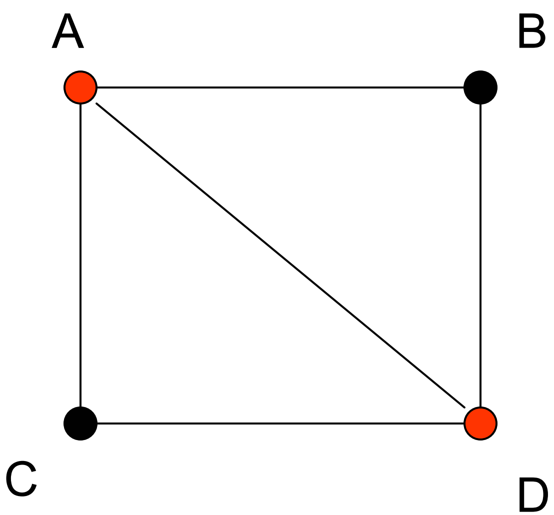

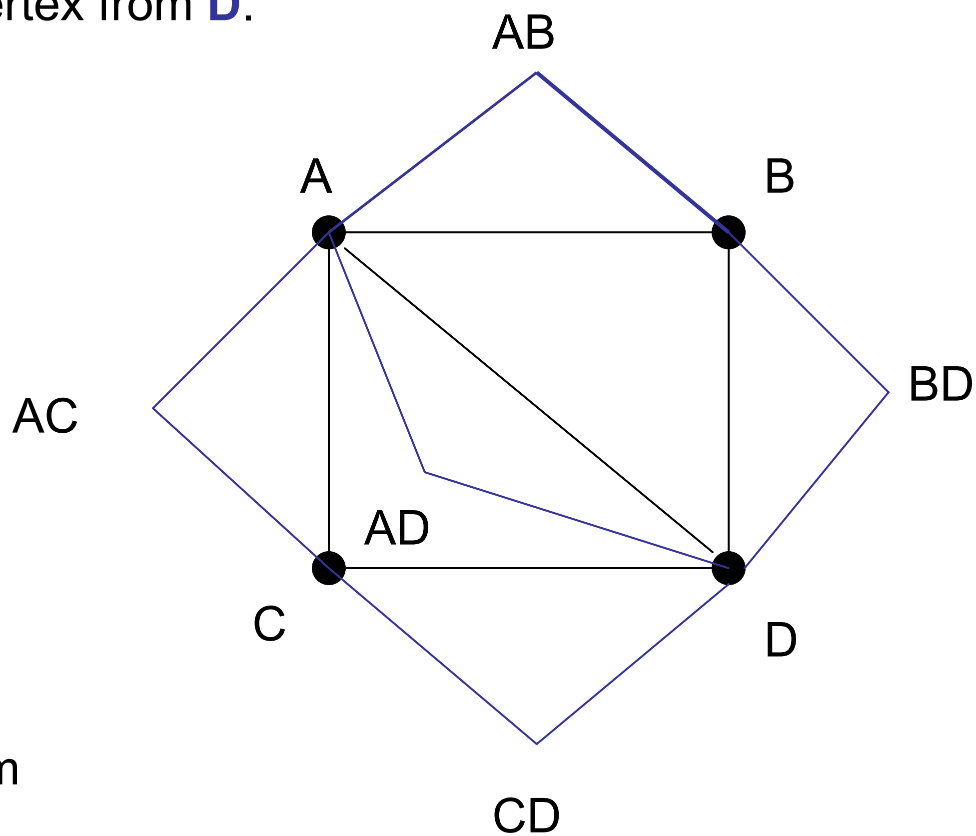

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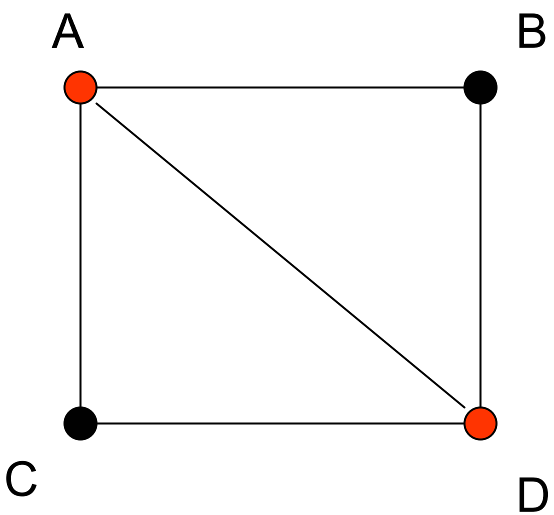

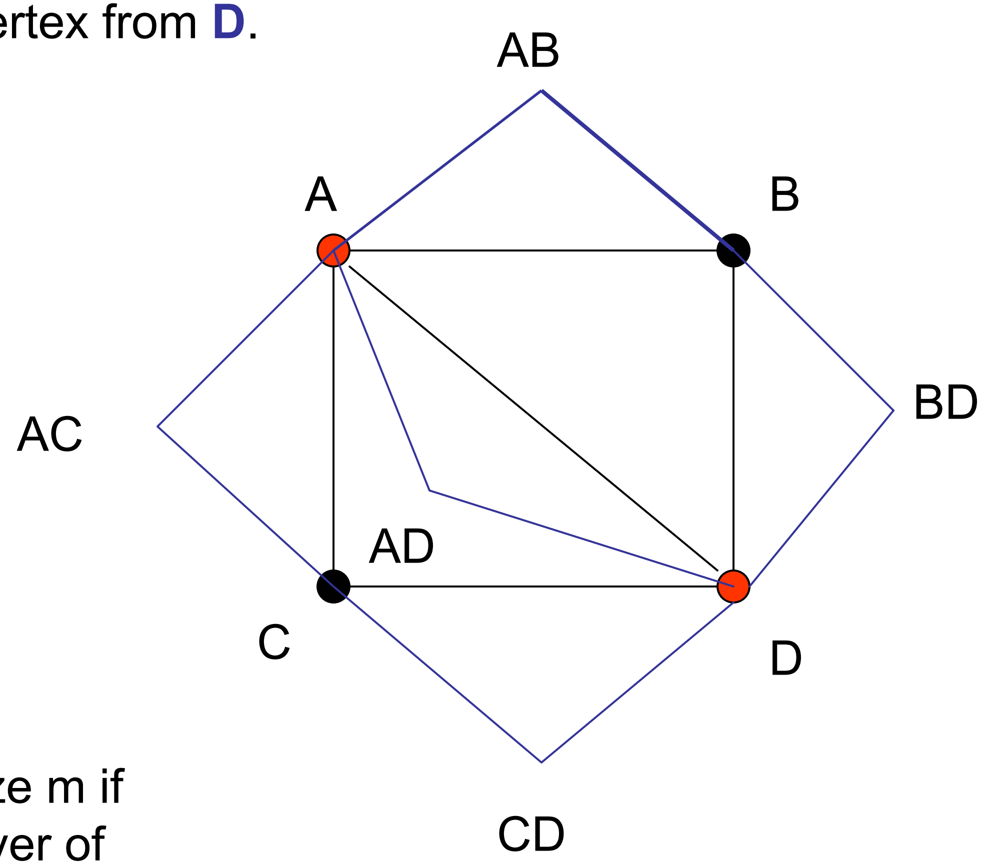

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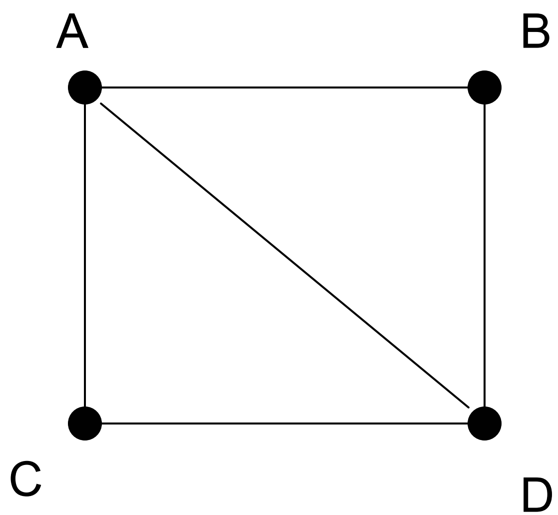

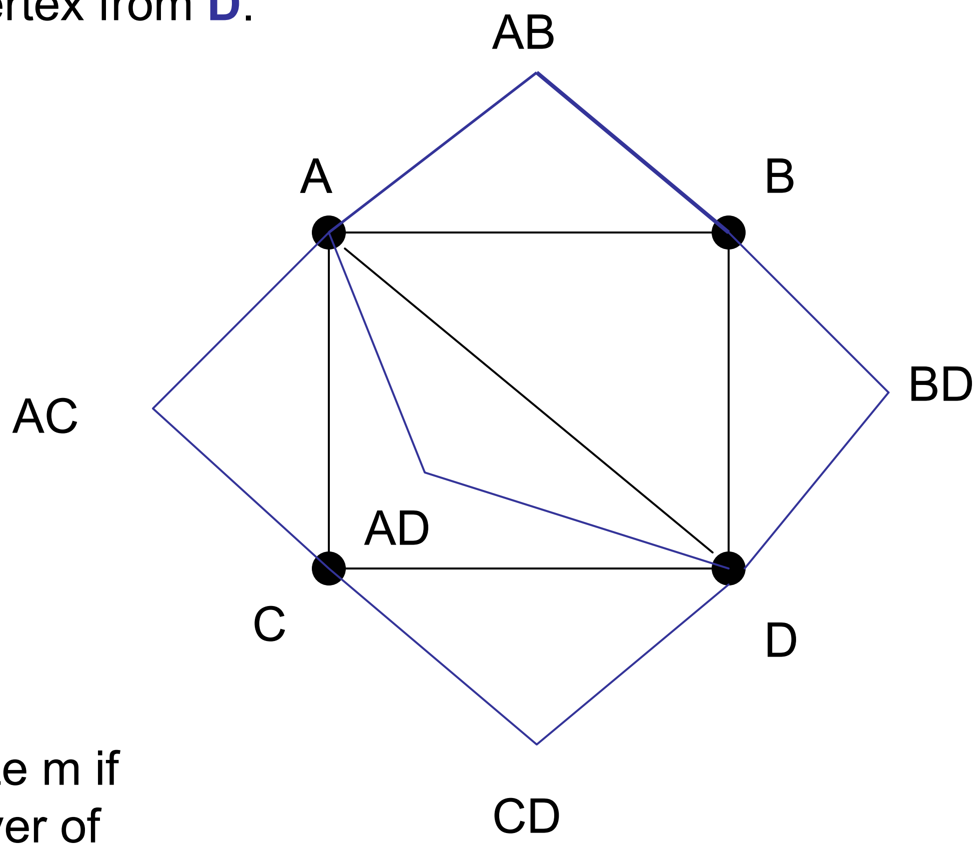

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Dealing with NP Complete problems

- Proving that a given problem is NP-Complete does not make the probem go away!! Udi Manber
- An NP-Complete problem cannot be solved precisely in polynomial time
  - We make compromises in terms of optimality, robustness, efficiency, or completeness of the solution.
- Approximation algorithms do not lead to optimal solutions
  - Probabilistic algorithms
  - Branch and bound
  - Backtracking

Backtracking

- The Hamiltonian Circuit Problem
- The Subset problem

Branch and Bound

- Job Assignment
- Knapsack